

ON THE POSITIVE SOLUTIONS OF SYSTEMS  
OF DIFFERENCE EQUATIONS

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**Abstract:** We show that every positive solutions of the following system of difference equations

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(2)}}, x_{n+1}^{(2)} = \frac{1}{x_n^{(3)}}, \dots, x_{n+1}^{(k)} = \frac{1}{x_n^{(1)}}$$

as well as of the system

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(k)}}, x_{n+1}^{(2)} = \frac{1}{x_n^{(1)}}, \dots, x_{n+1}^{(k)} = \frac{1}{x_n^{(k-1)}}$$

is periodic with period equal to  $2k$  if  $k \not\equiv 0 \pmod{2}$ , and with period  $k$  if  $k \equiv 0 \pmod{2}$ .

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### 1. Introduction

It is well known that all well defined solutions of the difference equation

$$x_{n+1} = \frac{1}{x_n} \tag{1.1}$$

are periodic with period two. Motivated by equation (1.1) we investigate

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the periodic character of the following two systems of difference equations, which can be considered as a natural generalizations of equation (1.1),

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(2)}}, x_{n+1}^{(2)} = \frac{1}{x_n^{(3)}}, \dots, x_{n+1}^{(k)} = \frac{1}{x_n^{(1)}} \quad (1.2)$$

and

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(k)}}, x_{n+1}^{(2)} = \frac{1}{x_n^{(1)}}, \dots, x_{n+1}^{(k)} = \frac{1}{x_n^{(k-1)}}. \quad (1.3)$$

In order to prove main results of the paper we need an auxiliary result which is contained in the following simple lemma from number theory. Let  $GCD(k, l)$  denote the greatest common divisor of the integers  $k$  and  $l$ .

**Lemma 1.** *Let  $k \in \mathbb{N}$ , and  $GCD(k, 2) = 1$ , then the numbers  $a_l = 2l + 1$ , (or  $a_l = -2l + 1$ ),  $l = 0, 1, \dots, k - 1$ , satisfy the following property  $a_{l_1} - a_{l_2} \not\equiv 0 \pmod{k}$ , when  $l_1 \neq l_2$ .*

*Proof.* Suppose the contrary, then we have

$$2(l_1 - l_2) = a_{l_1} - a_{l_2} = ks,$$

for some  $s \in \mathbb{Z} \setminus \{0\}$ .

Since  $GCD(k, 2) = 1$ , it follows that  $k$  is a divisor of  $l_1 - l_2$ . On the other hand, since  $l_1, l_2 \in \{0, 1, \dots, k - 1\}$ , we have  $|l_1 - l_2| < k$ , a contradiction.  $\square$

**Remark 1.** From Lemma 1 we see that the rests  $b_l$ ,  $l = 0, 1, \dots, k - 1$  of the numbers  $a_l = 2l + 1$ ,  $l = 0, 1, \dots, k - 1$ , obtained by dividing the numbers  $a_l$  by  $k$  are mutually different, they are contained in the set  $A = \{0, 1, \dots, k - 1\}$ , make a permutation of the ordered set  $(0, 1, \dots, k - 1)$ , and finally  $a_k = 2k + 1$  is the first number of the form  $2l + 1$ ,  $l \in \mathbb{N}$ , such that  $a_1 - a_0 \equiv 0 \pmod{k}$ .

## 2. Proofs of Main Results

In this section we formulate and prove the main results in this paper.

**Theorem 1.** *Consider equation (1.2), where  $k \geq 1$ . Then the following statements are true.*

(a) *If  $k \not\equiv 0 \pmod{2}$ , then every positive solution of equation (1.2) is periodic with period  $2k$ .*

(b) *If  $k \equiv 0 \pmod{2}$ , then every positive solution of equation (1.2) is periodic with period  $k$ .*

*Proof.* First note that the system is cyclic. Hence it is enough to prove that the sequence  $(x_n^{(1)})$  satisfies conditions (a) and (b) in the corresponding cases.

Further, note that for every  $s \in \mathbb{N}$  system (1.2) is equivalent to a system of  $ks$  difference equations of the same form, where

$$x_n^{(i)} = x_n^{(rk+i)}, \text{ for every } n \in \mathbb{N}, i \in \{1, \dots, k\} \text{ and } r = 0, 1, \dots, s - 1. \quad (2.1)$$

On the other hand, we have

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(2)}} = x_{n-1}^{(3)}. \quad (2.2)$$

(a) Let  $b_l, l = 0, 1, \dots, k - 1$ , are the rests mentioned in Remark 1. Then from (2.2) and Lemma 1 we obtain that

$$x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-3}^{(5)} = \dots = x_{n+1-2(k-1)}^{(2k-1)} = x_{n+1-2k}^{(2k+1)}. \quad (2.3)$$

Using (2.1) for sufficiently large  $s$ , we obtain that (2.3) is equivalent to (here we use the condition  $GCD(k, 2) = 1$ )

$$x_{n+1}^{(1)} = x_{n-1}^{(b_1)} = x_{n-3}^{(b_2)} = \dots = x_{n+1-2(k-1)}^{(b_{k-1})} = x_{n+1-2k}^{(1)}.$$

From this and since by Lemma 1 the numbers  $1, b_1, b_2, \dots, b_{k-1}$  are piecewise different, the result follows in this case.

(b) Let  $k = 2s$ , for some  $s \in \mathbb{N}$ . By (2.1) we have

$$x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-3}^{(5)} = \dots = x_{n+1-2s}^{(2s+1)} = x_{n+1-2s}^{(1)}$$

which yields the result. □

**Remark 2.** In order to make the proof of Theorem 1 clear to the reader, we explain what happens in the cases  $k = 2$  and  $k = 3$ .

For  $k = 2$  system (1.2) is equivalent to the system

$$\begin{aligned} x_{n+1}^{(1)} &= \frac{1}{x_n^{(2)}}, & x_{n+1}^{(2)} &= \frac{1}{x_n^{(3)}}, & x_{n+1}^{(3)} &= \frac{1}{x_n^{(4)}}, & x_{n+1}^{(4)} &= \frac{1}{x_n^{(5)}}, \\ x_{n+1}^{(5)} &= \frac{1}{x_n^{(6)}}, & x_{n+1}^{(6)} &= \frac{1}{x_n^{(1)}}, \end{aligned} \quad (2.4)$$

where we consider that  $x_n^{(1)} = x_n^{(3)} = x_n^{(5)}$  and  $x_n^{(2)} = x_n^{(4)} = x_n^{(6)}, n \in \mathbb{N}$ . From this and (2.2) we have  $x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-1}^{(5)}$  for every  $n \in \mathbb{N}$ . Using again

(2.2) we get  $x_{n+1}^{(1)} = x_{n-1}^{(1)}$ , which means that the sequence  $x_n^{(1)}$  is periodic with period equal to 2.

If  $k = 3$  system (1.2) is equivalent to system (2.4), where we consider that  $x_n^{(1)} = x_n^{(4)}$ ,  $x_n^{(2)} = x_n^{(5)}$  and  $x_n^{(3)} = x_n^{(6)}$ . Using this and (2.2) subsequently, it follows that

$$x_{n+1}^{(1)} = x_{n-1}^{(3)} = x_{n-3}^{(5)} = x_{n-3}^{(2)} = x_{n-5}^{(4)} = x_{n-5}^{(1)},$$

that is, the sequence  $x_n^{(1)}$  is periodic with period 6.

**Remark 3.** The fact that every positive solution of equation (1.1) is periodic with period two can be considered as the case  $k = 1$  in Theorem 1, that is, we can take that  $x_n^{(1)} = x_n^{(2)} = \dots = x_n^{(k)}$ , for all  $n \in \mathbb{N}$ .

Similarly to Theorem 1, using Lemma 2 with  $a_l = -2l + 1, l = 0, 1, \dots, k - 1$ , the following theorem can be proved.

**Theorem 2.** Consider equation (1.3), where  $k \geq 1$ . Then the following statements are true.

(a) If  $k \not\equiv 0 \pmod{2}$ , then every positive solution of equation (1.3) is periodic with period  $2k$ .

(b) If  $k \equiv 0 \pmod{2}$ , then every positive solution of equation (1.3) is periodic with period  $k$ .

*Proof.* First note that the system is cyclic. Hence it is enough to prove that the sequence  $(x_n^{(1)})$  satisfies conditions (a) and (b) in the corresponding cases.

Indeed, similarly to (2.2), we have

$$x_{n+1}^{(1)} = \frac{1}{x_n^{(k)}} = x_{n-1}^{(k-1)}. \tag{2.5}$$

(a) Let  $b_l, l = 0, 1, \dots, k - 1$ , are the rests mentioned in Remark 1. Then from (2.5) and Lemma 1 we obtain that

$$x_{n+1}^{(1)} = x_{n-1}^{(k-1)} = x_{n-3}^{(k-3)} = \dots = x_{n+1-2(k-1)}^{(-2k+3)} = x_{n+1-2k}^{(-2k+1)}. \tag{2.6}$$

Using (2.1) for sufficiently large  $s$ , we obtain that (2.6) is equivalent to (here we use the condition  $GCD(k, 2) = 1$ )

$$x_{n+1}^{(1)} = x_{n-1}^{(b_1)} = x_{n-3}^{(b_2)} = \dots = x_{n+1-2(k-1)}^{(b_{k-1})} = x_{n+1-2k}^{(1)}.$$

From this and since by Lemma 1 the numbers  $1, b_1, b_2, \dots, b_{k-1}$  are piecewise different, the result follows in this case.

(b) Let  $k = 2s$ , for some  $s \in \mathbb{N}$ . By (2.1) we have

$$x_{n+1}^{(1)} = x_{n-1}^{(k-1)} = x_{n-3}^{(k-3)} = \dots = x_{n+1-2s}^{(1)}$$

which yields the result. □

**Example 1.** Let  $k = 3$ . Then the solution of equation (1.2) with the initial conditions  $x_0 = p, y_0 = q$  and  $z_0 = r$ , in its interval of periodicity can be represented by the following table:

|       |               |     |               |     |               |     |               |     |               |     |               |     |
|-------|---------------|-----|---------------|-----|---------------|-----|---------------|-----|---------------|-----|---------------|-----|
| i     | 1             | 2   | 3             | 4   | 5             | 6   | 7             | 8   | 9             | 10  | 11            | 12  |
| $x_i$ | $\frac{1}{q}$ | $r$ | $\frac{1}{p}$ | $q$ | $\frac{1}{r}$ | $p$ | $\frac{1}{q}$ | $r$ | $\frac{1}{p}$ | $q$ | $\frac{1}{r}$ | $p$ |
| $y_i$ | $\frac{1}{r}$ | $p$ | $\frac{1}{q}$ | $r$ | $\frac{1}{p}$ | $q$ | $\frac{1}{r}$ | $p$ | $\frac{1}{q}$ | $r$ | $\frac{1}{p}$ | $q$ |
| $z_i$ | $\frac{1}{p}$ | $q$ | $\frac{1}{r}$ | $p$ | $\frac{1}{q}$ | $r$ | $\frac{1}{p}$ | $q$ | $\frac{1}{r}$ | $p$ | $\frac{1}{q}$ | $r$ |

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